Logarithmic diffusion and porous media equations: A unified description

I. T. Pedron,[1] R. S. Mendes,[2] T. J. Buratta,[2] L. C. Malacarne,[2] and E. K. Lenzi[2]
[1]*Universidade Estadual do Oeste do Paraná, Rua Pernambuco, 1777, 85960-000, Marechal Cândido Rondon, Paraná, Brazil*
[2]*Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo, 5790, 87020-900, Maringá, Paraná, Brazil*
(Received 7 April 2005; published 20 September 2005)

In this work we present the logarithmic diffusion equation as a limit case when the index that characterizes a nonlinear Fokker-Planck equation, in its diffusive term, goes to zero. A linear drift and a source term are considered in this equation. Its solution has a Lorentzian form, consequently this equation characterizes a superdiffusion like a Lévy kind. In addition an equation that unifies the porous media and the logarithmic diffusion equations, including a generalized diffusion equation in fractal dimension, is obtained. This unification is performed in the nonextensive thermostatistics context and increases the possibilities about the description of anomalous diffusive processes.

DOI: 10.1103/PhysRevE.72.031106 PACS number(s): 05.20.−y, 05.40.Fb, 05.40.Jc

I. INTRODUCTION

Diffusion is a very usual phenomena in nature and it happens, in general, when the system goes to an equilibrium state. Therefore it is of fundamental relevancy in physics, chemical, and biological processes. The linear dependence in the time growth of the mean square displacement $\langle x^2(t)\rangle \propto t$, or alternatively in the variance (when $\langle x\rangle \neq 0$), is the finger point of the Brownian movement and usual diffusion. It is a direct consequence of the central limit theorem and the Markovian nature of the underlying stochastic process [1]. In contrast, the anomalous diffusion is in general characterized by a nonlinear variance growth in the time, that is to say, the diffusion will be considered anomalous when the behavior goes out the former. This kind of diffusion has a fundamental role in the analysis of a large class of systems such as plasma diffusion [2], diffusion in turbulent fluids [3,4], fluids transportation in porous media [5], chaotic dynamics [6], non-Gaussian behavior of the heartbeat [7], diffusion on fractals [8–11], anomalous diffusion at liquid surfaces [12], in the study of vibrational energy in proteins [13], among other physical systems. In the anomalous behavior description the variance growth can be a power law kind $\langle(\Delta x)^2\rangle \propto t^{\eta}$, or even present another pattern. In this classification, when $\eta > 1$, we have a superdiffusive process, $\eta < 1$, a subdiffusive one, and $\eta = 1$ describes a usual diffusion. Furthermore, in an anomalous diffusive process, the variance can be not finite, describing a Lévy process, although it presents a well-defined index that characterizes such a process [14]. The description of this kind of process is based on the validity of the generalized central limit theorem, termed Lévy-Gnedenko, which states that, by N-fold convolution, a distribution with divergent lower moments tends to one of the Lévy stable class [15].

It is possible to simulate the anomalous behavior of the diffusion applying generalizations on the ordinary diffusion equation. It can be performed by introducing an appropriate time-dependence [16,17] or spatial dependence [8,9,18,19] in the equation's coefficients. Also we can apply fractional derivatives [20–23]. However, the introduction of nonlinearities reveals a large set of possibilities to describe anomalous diffusive processes. An interesting characteristic of the nonlinear Fokker-Planck equation is that its stationary solutions, and some particular time-dependent solutions, are such that maximize the Tsallis entropy, a nonextensive entropic form proposed in the last years by Tsallis [24,25]. This effort becomes necessary when the Boltzmann-Gibbs statistics fail, for instance, in the presence of long range interactions or memory effects and fractal phase space structure (see Ref. [26] for a recent review). The nonextensive mechanics statistics has shown a very fruitful scenario to study anomalous diffusion. In this way, several works dealing with anomalous diffusion were developed in such context. The connection between this formalism and the nonlinear Fokker-Planck equation was first pointed out by Plastino and Plastino [27]. The results were enlarged by Tsallis and Bukman [28], including a linear drift term. A phenomenological microscopic dynamics of the nonlinear Fokker-Planck equation is presented in [29], as well as a nonlinear Fokker-Planck equation with state-dependent diffusion [30]. A Tsallis maximum entropy solution of the nonlinear Fokker-Planck equation applied to the study of correlated anomalous diffusion [31,32], nonlinear fractional derivative Fokker-Planck like equation [22,23], aging in nonlinear diffusion [33], anomalous diffusion with absorption [34], and anomalous diffusion in a fractal dimension [18,19], are some examples among other references. In this direction, the generalized thermostatistic, based on the nonextensive Tsallis entropy, becomes a natural scenario as the correlated anomalous diffusion as the anomalous diffusionlike Lévy (variance is not finite) [35–38]. In this context, the index of nonlinearity joined to the nonlinear Fokker-Planck equation, in the correlated anomalous diffusion case, can be connected to the entropic index q of the Tsallis entropy and with the respective generalized distributions. This relation is just $q = 2-\nu$, where ν is the nonlinear index, and it is easy to verify that when $q = 2$ the diffusive term becomes trivial. We will show that this difficulty can be overcome by a logarithmic diffusion equation, or, in other words, this equation represents an alternative for the limit case where the exponent that characterizes the nonlinearity of the diffusive term goes to zero. In this way, the logarithmic diffusion equation is just inspired in nonlinear diffusion equations like Fokker-Planck [19,27,28]. We will show that the nonstationary solution for the logarithmic equation is a Lorentzian, and this implies that the second moment is not finite. It indicates that this diffusion equation is related to superdiffusive processes, of the Lévy kind. On the other hand, in the nonextensive statistical mechanics scenario it is possible to unify the correlated anomalous diffusion equation, or porous media equation, and the logarithmic diffusion equation. Thus we enlarge the description of diffusive processes by a unified equation that congregates the correlated anomalous diffusion and a Lévy like one.

This work is structured as follows. In Sec. II we will present the logarithmic diffusion equation and its exact solution. In Sec. III this equation is solved including a linear drift term. In the following section it is solved when an absorption term is present. We will obtain in Sec. V the stationary solution, and in Sec. VI we will perform the unification of the porous media and logarithmic diffusion equations. The connection with the diffusion on fractals is included. Finally, in the last section we will present the conclusion and final remarks.

II. LOGARITHMIC DIFFUSION EQUATION

To motivate our discussion, we will first consider the porous media equation,

$$\frac{\partial\rho}{\partial t} = D\nabla^2\rho^{\nu}. \tag{2.1}$$

It has been employed in the analysis of percolation of gases through porous media ($\nu \geqslant 2$) [39], thin saturated regions in porous media ($\nu=2$) [40], a standard solid-on-solid model for surface growth ($\nu=3$), thin liquid films spreading under gravity ($\nu=4$) [41], among others [42]. A solution of Eq. (2.1) is the generalized q-Gaussian [28],

$$\rho(r,t) = \frac{1}{Z(t)}[1-(1-q)\beta(t)r^2]^{1/(1-q)}. \tag{2.2}$$

By direct substitution of Eq. (2.2) into Eq. (2.1) it is easy to verify that

$$\frac{d\beta}{dt} = -4D\nu\beta^2 Z^{q-1}, \tag{2.3}$$

$$\frac{1}{Z}\frac{dZ}{dt} = 2D\beta\nu Z^{q-1}, \tag{2.4}$$

and we obtain $\beta(t) \propto t^{-2/(3-q)}$ and $Z(t) \propto t^{1/(3-q)}$. Of course, when $q \to 1$ we recover the results for the usual diffusion. This solution is obtained taking into account the relation $q=2-\nu$. And what happens when $q=2$? It is obvious that in this case the right side of Eq. (2.1) vanishes all the time and Eq. (2.2) is not the solution now. However, Eq. (2.2), for $q=2$, recovers a very important function: the Lorentzian. A question arises: what would be the shape of the nonlinear diffusion equation whose solution was a Lorentzian?

It is known that $\ln x$ decreases more slowly than any power x^r, when r goes to zero positively, so we are influenced to substitute ρ^{ν} by $\ln\rho$ in Eq. (2.1) when $\nu \to 0$. In this perspective, we introduce the logarithmic diffusion equation

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2 \ln\rho}{\partial x^2}. \tag{2.5}$$

This equation emerges in plasma physics [43] and, in particular, has been predicted for cross-field convective diffusion of plasma including mirror effects [44]. The same equation describes the expansion of a thermalized electron cloud [45] and also arises in studies of the central limit approximation to Carleman's model of the Boltzmann equation [46,47].

It is verified directly that Eq. (2.5) presents the Lorentzian solution

$$\rho(x,t) = \frac{1}{Z(t)}\frac{1}{[1+\beta(t)x^2]}. \tag{2.6}$$

The substitution of Eq. (2.6) in Eq. (2.5) shows that $\beta(t)$ and $Z(t)$ obey the equations

$$\frac{1}{Z^2}\frac{dZ}{dt} = 2D\beta, \tag{2.7}$$

$$\frac{1}{Z^2}\frac{dZ}{dt} + \frac{1}{\beta Z}\frac{d\beta}{dt} = -2D\beta. \tag{2.8}$$

They can be decoupled taking account of the relation $Z\beta^{1/2}=Z_0\beta_0^{1/2}$ which is valid all the time. The solutions are

$$\beta(t) = \beta_0(1+2D\beta_0 Z_0 t)^{-2} \tag{2.9}$$

and

$$Z(t) = Z_0(1+2D\beta_0 Z_0 t). \tag{2.10}$$

It is interesting to point out that for the Lorentzian the variance is not finite. This fact, characteristic of the Lévy distributions, indicates that the logarithmic equation is associated to superdiffusive regimes. In fact, a dimensional analysis of the logarithmic diffusion equation (2.5) and the Lévy diffusion equation $\partial\rho/\partial t = D\partial^{\mu}\rho/\partial|x|^{\mu}$, with $\mu=1$, leads to the common ballistic behavior in the sense that x scales as t. Moreover, Eq. (2.5) is a particular $\mu=1$ case of

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\rho^{-\mu}\frac{\partial\rho}{\partial x}, \quad 0<\mu<2. \tag{2.11}$$

When the coefficient diffusion $D(\rho)$ (or the thermal coefficient of conductivity) can be approximated as $\rho^{-\mu}$, its divergence for small ρ causes a much faster spread of mass (or heat) than in the linear case ($\mu=0$), justifying the terminology superfast diffusion to these processes [48].

For the N-dimensional case, Eq. (2.5) assumes the shape

$$\frac{\partial\rho}{\partial t} = D\nabla^2\ln\rho, \tag{2.12}$$

with $\nabla^2=\Sigma_{i=1}^{N}\partial^2/\partial x_i^2$ and whose Lorentzian solution, $\rho(|\mathbf{x}|,t)$, presents

$$\beta(t) = \beta_0[1-2(N-2)D\beta_0 Z_0 t]^{2/(N-2)} \tag{2.13}$$

and

$$Z(t) = Z_0[1-2(N-2)D\beta_0 Z_0 t]^{N/(2-N)}. \tag{2.14}$$

III. PRESENCE OF EXTERNAL FORCES

Now we analyze Eq. (2.5) describing a process that includes a linear drift term (external force) $F(x)$,

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2}{\partial x^2}\ln\rho - \frac{\partial}{\partial x}(F\rho). \tag{3.1}$$

For the case $F(x)=k_1-k_2x$, the solution of Eq. (3.1) is the drifted Lorentzian

$$\rho(x,t) = \frac{1}{Z(t)}\frac{1}{\{1+\beta(t)[x-x_0(t)]^2\}}. \tag{3.2}$$

In fact, $\beta(t)$, $Z(t)$, and $x_0(t)$ obey the equations:

$$\frac{dx_0}{dt} = k_1 - k_2x_0, \tag{3.3}$$

$$\frac{1}{\beta}\frac{d\beta}{dt} = -4D\beta Z + 2k_2, \tag{3.4}$$

$$\frac{1}{Z}\frac{dZ}{dt} = 2D\beta Z - k_2. \tag{3.5}$$

Equation (3.3) does not depend on the index related to the nonlinearity, therefore it arises not only in the usual diffusion equation but also in the porous media ones ($\nu\neq 1$). Thus the solution of Eq. (3.3) is

$$x_0(t) = \left[x_0(0) + \frac{k_1}{k_2}(e^{k_2t}-1)\right]e^{-k_2t}. \tag{3.6}$$

In this turn, the solutions of Eqs. (3.4) and (3.5) have the form

$$\beta(t) = \beta_0[1-g(t)]^{-2} \tag{3.7}$$

and

$$Z(t) = Z_0[1-g(t)], \tag{3.8}$$

where

$$g(t) = \left[\frac{(k_2-2D\beta_0Z_0)}{k_2}(e^{k_2t}-1)\right]e^{-k_2t} \tag{3.9}$$

with $g(0)=0$.

IV. LOGARITHMIC EQUATION WITH SOURCE TERM

We can consider the presence of a time-dependent source term. In this case, Eq. (2.5) is written as

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2}{\partial x^2}\ln\rho - \alpha(t)\rho. \tag{4.1}$$

The source term in this equation can be removed by an appropriate change in the solution

$$\rho(\mathbf{r},t) = e^{[-\int_0^t\alpha(t')dt']}\hat{\rho}(\mathbf{r},t) \tag{4.2}$$

and we rewrite the time variable. Thus, the solution of Eq. (4.1) is

$$\rho(x,t) = \frac{1}{Z(\tau(t))[1+\beta(\tau(t))x^2]}\exp\left(-\int_0^t\alpha(t')dt'\right),$$

with $\beta(\tau(t))$ and $Z(\tau(t))$ being of the form (2.9) and (2.10), where $D=1$ and t is replaced by $\tau(t)=\int_0^t\tilde{D}(t')dt'$, with $\tilde{D}(t)=D\exp(\int_0^t\alpha(t')dt')$. We can to extend this solution for the N-dimensional case by employing Eqs. (2.13) and (2.14).

V. STATIONARY CASE

We define now the probability density current

$$J = F\rho - D\frac{\partial\ln\rho}{\partial x} \tag{5.1}$$

so that Eq. (3.1) represents a continuity equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0. \tag{5.2}$$

In the stationary case, $dJ/dx=0$, and this implies $J=cte=0$, since we are applying the condition that the current vanishes at infinity. This fact permits us to write, considering $F=-dV/dx$,

$$\frac{d\ln\rho}{dx} = -\frac{1}{D}\frac{dV}{dx}\rho \tag{5.3}$$

whose solution is

$$\rho = \frac{1}{1+\beta V}, \tag{5.4}$$

where it was assumed $V(0)=0$, $\beta=\rho_0/D$, and $\rho(0)=\rho_0=1$. Note that the structure (5.4) is preserved in the N-dimensional case.

The solution (5.4) is the $q=2$ case of the generalized exponential

$$\rho = [1-(1-q)\beta V]^{1/(1-q)}, \tag{5.5}$$

which, in this turn, recovers the Boltzmann distribution in the limit $q\to 1$. Furthermore, we remind that the Boltzmann distribution is the stationary solution of the usual diffusion equation, so Eq. (5.4) is analogous for the logarithmic equation.

VI. UNIFICATION OF POROUS MEDIA AND LOGARITHMIC EQUATIONS

The unification of Eqs. (2.1) and (2.12) will be accomplished by using the generalized logarithmic function, the q-logarithm, defined as

$$\ln_q x = \frac{x^{1-q}-1}{1-q}. \tag{6.1}$$

In this way, the unified equation proposed in this work presents the following structure:

$$\frac{\partial\rho}{\partial t} = \bar{D}\nabla^2\ln_{q-1}\rho, \tag{6.2}$$

and it is clear that, when $q\to 2$, the function $\ln_1$ recovers the logarithmic one. When $q=2-\nu\neq 2$, we have $\ln_{q-1}=(\rho^\nu$

$-1)/\nu$ and Eq. (6.2) reobtains the porous media equation (2.1) with $D=\bar{D}/\nu$.

We can consider the nonlinear diffusion equation with radial symmetry, taking account of the spatial-dependence in the diffusion coefficient, $r^{-\theta}$, and a nonintegral dimension d [18],

$$\frac{\partial \rho}{\partial t} = D\tilde{\Delta}\rho^{\nu}, \tag{6.3}$$

with

$$\tilde{\Delta} \equiv r^{-(d-1)}\frac{\partial}{\partial r}r^{d-1-\theta}\frac{\partial}{\partial r}. \tag{6.4}$$

With the operator $\tilde{\Delta}$ written like this there are no restrictions for the possible values for d. In this way, d can be interpreted as a fractal dimension in an embedding N-dimensional space. Equation (6.3), in the $\nu=1$ case, recovers the diffusion equation introduced in Refs. [8,9]. Analogously to Eq. (6.2), we enlarge the applicability domain of Eq. (6.3) and we have

$$\frac{\partial \rho}{\partial t} = \bar{D}\tilde{\Delta}\ln_{q-1}\rho, \tag{6.5}$$

with $\tilde{\Delta}$ defined by Eq. (6.4). The *ansatz*

$$\rho(r,t) = \frac{1}{Z(t)}[1-(1-q)\beta(t)r^{\lambda}]^{1/(1-q)}, \tag{6.6}$$

with $\lambda=2+\theta$, remains valid and substituted in Eq. (6.5) conducts to the equations

$$\frac{dZ(t)}{dt} = \bar{D}\lambda d\beta(t)Z^{q}(t),$$

$$\frac{d\beta(t)}{dt} = -\bar{D}\lambda^{2}\beta^{2}(t)Z^{q-1}(t). \tag{6.7}$$

Such equations are decoupled and solved and their solutions are

$$\beta(t) = \beta_0[1+At]^{-\lambda/[\lambda+d(1-q)]} \tag{6.8}$$

and

$$Z(t) = Z_0[1+At]^{d/[\lambda+d(1-q)]} \tag{6.9}$$

with

$$A = \bar{D}\lambda[\lambda+d(1-q)]\beta_0 Z_0^{q-1}, \tag{6.10}$$

$\beta_0=\beta(0)$ and $Z_0=Z(0)$. Thus we obtain the stretched Lorentzian if $\lambda<2$ and short if $\lambda>2$. In fact, for $q=2$, the solution is

$$\rho(r,t) = \frac{1}{Z(t)}\left[\frac{1}{1+\beta(t)r^{\lambda}}\right], \tag{6.11}$$

where $\beta(t)=\beta_0[1+At]^{-\lambda/[\lambda-d]}$ and $Z(t)=Z_0[1+At]^{d/[\lambda-d]}$. Observe that when $\theta=0$ and $d=N=1$ we obtain the results (2.2) for $q\neq 2$, and Eq. (2.6) for $q=2$.

We pointed out that the factor $(2-q)$ implicated Eqs. (2.1) and (6.3) became trivial at the value $q=2$. With our strategy we eliminate this undesirable behavior. From the performed generalization in this section, we can conjecture a nonlinear diffusion equation, that presents ρ^{ν}, can be extended to $\nu=0$, substituting ρ^{ν} by $\ln_{q-1}\rho$, with $q=2-\nu$. Thus, in a general way, we can unify all the equations presented in this work by the equation

$$\frac{\partial \rho}{\partial t} = \sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(D_{ij}\frac{\partial}{\partial x_j}\ln_{q-1}\rho\right) - \sum_{i=1}^{N}\frac{\partial}{\partial x_i}(f_i\rho) - \alpha(t)\rho. \tag{6.12}$$

VII. CONCLUSION

In this work we enlarged the application domain of the porous media equation for the $\nu=0$ case and we presented the logarithmic diffusion equation as an alternative procedure to this limit case. Its solution has a Lorentzian form and it can characterize superdiffusive processes, of Lévy kind. A unification of porous media and logarithmic diffusion equation is obtained, and in a more general form, with the fractal nonlinear diffusion equation. This unified description is performed in the nonextensive statistical mechanics scenario. This accomplishment represents progress in the formal description of diffusive processes and in its solutions as well. In this direction, the final proposed equation interpolates others with a consecrated position in the literature. It is desirable that the equations presented in this work, or in particular special cases of them, show physics situations in which there is competition among different mechanisms that generate anomalous diffusion.

[1] S. K. Ma, *Statistical Mechanics* (World Scientific, Singapore, 1985).
[2] J. G. Berryman, J. Math. Phys. **18**, 2108 (1977).
[3] M. F. Shlesinger, J. Klafter, and B. J. West, Physica A **140**, 212 (1986).
[4] M. F. Shlesinger, B. J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).
[5] H. Spohn, J. Phys. I **3**, 69 (1993).
[6] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).
[7] C.-K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. **70**, 1343 (1993).
[8] B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **54**, 455 (1985).
[9] B. O'Shaughnessy and I. Procaccia, Phys. Rev. A **32**, 3073 (1985).
[10] J. Stephenson, Physica A **222**, 234 (1995).
[11] R. Metzler, W. G. Glockle, and T. F. Nonnenmacher, Physica

A **211**, 13 (1994).
[12] O. V. Bychuk and B. O'Shaughnessy, Phys. Rev. Lett. **74**, 1795 (1995).
[13] X. Yu and D. M. Leitner, J. Chem. Phys. **119**, 12673 (2003).
[14] J. P. Bouchad and A. Georges, Phys. Rep. **195**, 127 (1990).
[15] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Cambridge, MA, 1954).
[16] F. Lillo and R. N. Mantegna, Phys. Rev. E **61**, R4675 (2000).
[17] L. C. Malacarne, R. S. Mendes, I. T. Pedron, and E. K. Lenzi, Phys. Rev. E **65**, 052101 (2002).
[18] L. C. Malacarne, R. S. Mendes, I. T. Pedron, and E. K. Lenzi, Phys. Rev. E **63**, 030101(R) (2001).
[19] I. T. Pedron, R. S. Mendes, L. C. Malacarne, and E. K. Lenzi, Phys. Rev. E **65**, 041108 (2002).
[20] A. Compte, Phys. Rev. E **53**, 4191 (1996).
[21] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[22] M. Bologna, C. Tsallis, and P. Grigolini, Phys. Rev. E **62**, 2213 (2000).
[23] E. K. Lenzi, R. S. Mendes, L. C. Malacarne, and I. T. Pedron, Physica A **319**, 245 (2003).
[24] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[25] E. M. F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); Corrigenda: **24**, 3187 (1991); **25**, 1019 (1992).
[26] *Nonextensive Statistical Mechanics and its Applications*, edited by S. Abe and Y. Okamoto, Vol. 560 of *Lecture Notes in Physics* (Springer-Verlag, Berlin, 2001); *Nonextensive Entropy-interdisciplinary Applications*, edited by M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004); an updated bibliography can be found at the web site http://tsallis.cat.cbpf.br/biblio.htm.
[27] A. R. Plastino and A. Plastino, Physica A **222**, 347 (1995).
[28] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[29] L. Borland, Phys. Rev. E **57**, 6634 (1998).
[30] L. Borland, F. Pennini, A. R. Plastino, and A. Plastino, Eur. Phys. J. B **12**, 285 (1999).
[31] A. Compte and D. Jou, J. Phys. A **29**, 4321 (1996).
[32] A. Compte, D. Jou, and Y. Katayama, J. Phys. A **30**, 1023 (1997).
[33] D. A. Stariolo, Phys. Rev. E **55**, 4806 (1997).
[34] G. Drazer, H. S. Wio, and C. Tsallis, Phys. Rev. E **61**, 1417 (2000).
[35] P. A. Alemany and D. H. Zanette, Phys. Rev. E **49**, R956 (1994).
[36] D. H. Zanette and P. A. Alemany, Phys. Rev. Lett. **75**, 366 (1995).
[37] C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995).
[38] D. Prato and C. Tsallis, Phys. Rev. E **60**, 2398 (1999).
[39] M. Muskat, *The Flow of Homogeneous Fluid through Porous Media* (McGraw-Hill, New York, 1937).
[40] P. Y. Polubarinova-Kochina, *Theory of Ground Water Movement* (Princeton University Press, Princeton, NJ, 1962).
[41] J. Buckmaster, J. Fluid Mech. **81**, 735 (1984).
[42] W. L. Kath, Physica D **12**, 375 (1984).
[43] J. G. Berryman and C. J. Holland, J. Math. Phys. **23**, 983 (1982).
[44] T. Kamimura and J. M. Dawson, Phys. Rev. Lett. **36**, 313 (1976).
[45] K. E. Lonngren and A. Hirose, Phys. Lett. A **59**, 285 (1976).
[46] T. G. Kurtz, Trans. Am. Math. Soc. **186**, 259 (1973).
[47] H. P. McKean, Isr. J. Math. **21**, 54 (1975).
[48] P. Rosenau, Phys. Rev. Lett. **74**, 1056 (1995).